

# ON THE NUMERICAL RADIUS OF THE TRUNCATED ADJOINT SHIFT

HAYKEL GAAYA

**ABSTRACT.** A celebrated theorem of Fejer (1915) asserts that for a given positive trigonometric polynomial  $\sum_{j=-n+1}^{n-1} c_j e^{ijt}$ , we have  $|c_1| \leq c_0 \cos \frac{\pi}{n+1}$ . A more recent inequality due to U. Haagerup and P. de la Harpe [9] asserts that, for any contraction  $T$  such that  $T^n = 0$ , for some  $n \geq 2$ , the inequality  $\omega_2(T) \leq \cos \frac{\pi}{n+1}$  holds, and  $\omega_2(T) = \cos \frac{\pi}{n+1}$  when  $T$  is unitarily equivalent to the extremal operator  $S_n^* = S^*|_{\mathbb{C}^n} = S^*|_{\text{Ker}(u_n(S^*))}$  where  $u_n(z) = z^n$  and  $S^*$  is the adjoint of the shift operator on the Hilbert space of all square summable sequences. Apparently there is no relationship between them. In this mathematical note, we show that there is a connection between Taylor coefficients of positive rational functions on the torus and numerical radius of the extremal operator  $S^*(\phi) = S^*|_{\text{Ker}(\phi(S^*))}$  for a precise inner function  $\phi$ . This result completes a line of investigation begun in 2002 by C. Badea and G. Cassier [1]. An upper and lower bound of the numerical radius of  $S^*(\phi)$  are given where  $\phi$  is a finite Blaschke product with unique zero.

## 1. INTRODUCTION

Let  $\mathcal{H}$  be a complex separable Hilbert space and  $\mathcal{B}(\mathcal{H})$  the collection of all bounded linear operators on  $\mathcal{H}$ . The numerical range of an operators  $T$  in  $\mathcal{B}(\mathcal{H})$  is the subset

$$W(T) = \{ \langle Tx, x \rangle \in \mathbb{C}; x \in \mathcal{H}, \|x\| = 1 \}$$

of the plane, where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathcal{H}$  and the numerical radius of  $T$  is defined by

$$\omega_2(T) = \sup \{ |z|; z \in W(T) \}.$$

$\mathcal{Re}(T)$  is the self-adjoint operator defined by

$$\mathcal{Re}(T) = \frac{1}{2}(T + T^*).$$

We denote by  $[x]$  the integer part of  $x$  and by  $S$  the unilateral shift acting on the Hardy space  $\mathbb{H}^2$  of the square summable analytic functions and by  $S^*$  its adjoint:

$$\begin{array}{rcl} S & : & \mathbb{H}^2 \rightarrow \mathbb{H}^2 \\ f & \mapsto & zf(z) \end{array}$$

$$\begin{array}{rcl} S^* & : & \mathbb{H}^2 \rightarrow \mathbb{H}^2 \\ f & \mapsto & \frac{f(z) - f(0)}{z} . \end{array}$$

---

2000 *Mathematics Subject Classification.* 47A12, 47B35.

*Key words and phrases.* Numerical radius, Numerical range, Truncated shift, Eigenvalues, Toeplitz forms, Inequalities for positive trigonometric polynomials.

Beurling's theorem implies that the non zero invariant subspaces of  $S$  are of the forme  $\phi \mathbb{H}^2$ , where  $\phi$  is some inner function . Let  $S(\phi)$  denote the compression of  $S$  to the space  $H(\phi) = \mathbb{H}^2 \ominus \phi \mathbb{H}^2$  :

$$S(\phi)f(z) = P(zf(z)),$$

where  $P$  denotes the ortogonal projection from  $\mathbb{H}^2$  onto  $H(\phi)$ . We denote by  $S^*(\phi)$  the adjoint of  $S(\phi)$ :

$$S^*(\phi) = S(\phi)^* = S^*|_{H(\phi)} = S^*|_{\text{Ker}(\phi(S)^*)} .$$

The model operator  $S(\phi)$  has many properties and it was studied intensively in the 1960s and '70s. For example, it has norm 1 (for  $\dim H(\phi) > 1$ ) and it is cyclic. The function  $\phi$  is the minimal function of  $S(\phi)$  meaning that  $\phi(S(\phi)) = 0$  and  $\phi$  divides any function  $\psi$  in  $H^\infty$  with  $\psi(S(\phi)) = 0$ . The space  $H(\phi)$  is finite-dimensional exactly when  $\phi$  is a finite Blaschke product :

$$\phi(z) = \prod_{j=1}^n \frac{z - \alpha_j}{1 - \overline{\alpha_j}z}.$$

In this case the polynomial  $p(z) = \prod_{j=1}^n (z - \alpha_j)$  is both the minimal and characteristic polynomial of  $S(\phi)$  and  $(\alpha_j)_{1 \leq j \leq n}$  are its eigenvalues. In particular, if  $\phi(z) = z^n$  then  $S(\phi)$  is unitarily equivalent to  $S_n$  where

$$S_n = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

In 1992 U. Haagerup and P. de la Harpe proved that  $W(S_n)$  is the disc  $D_n = \left\{ z \in \mathbb{C}; |z| \leq \cos \frac{\pi}{n+1} \right\}$  and

$$(1.1) \quad \omega_2(S_n) = \cos \frac{\pi}{n+1}$$

and more generally a natural connection between Fejer's inequality and the numerical radius of a nilpotent matrix was established by Haagerup and de la Harpe. They proved, using solely elementary methods (positive definite kernels) that:

**Theorem 1.1** ([9]). *Let  $T$  be an operator on  $\mathcal{H}$  such that  $T^n = 0$  for some  $n \geq 2$ . One has:*

$$\omega_2(T) \leq \|T\| \cos \frac{\pi}{n+1}$$

and  $\omega_2(T) = \|T\| \cos \frac{\pi}{n+1}$  when  $T$  is unitarily equivalent to  $\|T\|S_n$ .

The reader may consult [10] chapter 22 for properties of numerical ranges of operators in general, [11] chapter 1 for those of finite dimensional operators and particulary [6] for the geometric properties of the numerical range of  $S(\phi)$ .

#### Organisation of the paper:

In [1], C. Badea and G. Cassier showed:

**Theorem 1.2.** ([1]) *Let  $F = P/Q$  be a rational function with no principal part ( $d^\circ P < d^\circ Q$ ) which is positive on the torus. Then the Taylor coefficient  $c_k$  of order  $k$  satisfies the following inequality*

$$|c_k| \leq c_0 \omega_2(R^k),$$

where  $R = S^*|_{\text{Ker}(Q(S^*))}$ .

In Section 2, our main theorem is the Theorem 1.2. We give an extension of the result of C. Badea and G. Cassier for Taylor coefficients of all rational functions which are positive on the torus. We make no extra assumptions about  $P$  and  $Q$ . We do not, for exemple require them to obey any degree restrictions, they need only be coprime. The theorem has many applications and will explain how we can easily recover the remarkable Egerváry and Szász inequality. See Corollary 2.3.

Toeplitz matrices are found in several areas of mathematics and its applications such as complex and harmonic analysis. The KMS Toeplitz matrix

$$K_n(\alpha) = (\alpha^{|r-s|})_{r,s=1}^n$$

associated with the Poisson kernel introduced by Kac, Murdoh and Szegő [12] is of particular interest in these areas. In order to formulate our problem we first review in Section 3.1 some of the known results on the spectra of these matrices and we give a better upper bound of  $\lambda_1^{(n)}$ , where  $\lambda_1^{(n)}$  is the largest eigenvalue of the KMS matrix. In Section 3.2, our main theorem is the Theorem 3.7 which gives an upper and lower bound of the numerical radius of the truncated shift  $S^*(\phi)$  where  $\phi(z) = \left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)^n$  with  $0 \leq \alpha < 1$  is a finite Blaschke product with unique zero. Our preoccupation with this particular case  $0 \leq \alpha < 1$  to the exclusion to any  $\alpha$  in the disc is explained by the fact that the numerical radius is independent with the argument of  $\alpha$ . This formula is expressed in terms of eigenvalues of the KMS matrices and provides an easy proof for the Haagerup and de la Harpe result (1.1).

## 2. MAIN THEOREM

There are many classical inequalities for coefficients of positive trigonometric polynomials. The next result shows the links between the numerical radius of the truncated adjoint shift and the Taylor coefficients of rational functions positive on the torus.

**Theorem 2.1.** *Let  $F = P/Q$  be a rational function which is positive on the torus, where  $P$  and  $Q$  are coprime. Denote by*

$$\phi(z) = \prod_{j=1}^p \left( \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \right)^{m_j}$$

and

$$\psi(z) = \prod_{j=1}^q \left( \frac{z - \beta_j}{1 - \bar{\beta}_j z} \right)^{d_j}$$

the respectively finite Blaschke products formed by the nonzero roots of  $P$  and  $Q$  in the open disc, let  $m = \sum_{j=1}^p m_j$  and  $d = \sum_{j=1}^q d_j$ . Then the Taylor coefficient  $c_k$  of order  $k$  of  $F$  satisfies the following inequality:

$$|c_k| \leq c_0 \omega_2(S^{*k}(\varphi)), \text{ where } \varphi(z) = z^{\max(0, m-d+1)} \psi(z).$$

**Lemma 2.2** ([1] lemma 3.2). *Let  $u$  be a inner function and let  $f$  be a positive function in the subspace  $\overline{u} \mathbb{H}_0^1$  of  $L^1(\mathbb{T})$ . Then there exists a function  $h$  in  $H(u) = \mathbb{H}^2 \ominus u \mathbb{H}^2$  such that  $f = |h|^2$ .*

*Proof.* First, note that by continuity we may assume that  $F$  is strictly positive on the torus. Let  $F = P/Q$  and assume that  $F(z) > 0$  for every  $z \in \mathbb{T}$ . Now, let

$$G(z) = \overline{F\left(\frac{1}{\bar{z}}\right)}.$$

We see that  $G$  is analytic, except a finite set of complex numbers. Since  $F$  is real on the torus, we have  $G(e^{it}) = \overline{F(e^{it})} = F(e^{it})$  for every  $t \in \mathbb{R}$  and the analytic extension principle implies that

$$F(z) = \frac{P(z)}{Q(z)} = G(z) = \frac{\overline{P\left(\frac{1}{\bar{z}}\right)}}{\overline{Q\left(\frac{1}{\bar{z}}\right)}},$$

thus

$$(2.1) \quad P(z) \overline{Q\left(\frac{1}{\bar{z}}\right)} = \overline{P\left(\frac{1}{\bar{z}}\right)} Q(z)$$

except for a finite set in  $\mathbb{C}$ . (2.1) implies that if  $P(\alpha) = 0$ , with  $\alpha \neq 0$  then necessarily  $P\left(\frac{1}{\bar{\alpha}}\right) = 0$ . Then  $P$  can be written as

$$P(z) = c_1 z^{m_0} (z - \alpha_1)^{m_1} \dots (z - \alpha_p)^{m_p} (1 - \overline{\alpha_1} z)^{m_1} \dots (1 - \overline{\alpha_p} z)^{m_p}$$

with a constant  $c_1$ . With the same argument as before,  $Q$  can be written as

$$Q(z) = c_2 z^{d_0} (z - \beta_1)^{d_1} \dots (z - \beta_q)^{d_q} (1 - \overline{\beta_1} z)^{d_1} \dots (1 - \overline{\beta_q} z)^{d_q}$$

with a constant  $c_2$ . Since  $P$  and  $Q$  are coprime, we must have  $m_0 = 0$  or  $d_0 = 0$ .

Then

$$F(e^{it}) = |F(e^{it})| = c \left| \frac{P_1(e^{it})}{Q_1(e^{it})} \right|^2$$

where  $P_1(z) = \prod_{j=1}^p (z - \alpha_j)^{m_j}$  and  $Q_1(z) = \prod_{j=1}^q (z - \beta_j)^{d_j}$  therefore we have

$$F(e^{it}) = c \frac{\prod_{j=1}^p (e^{it} - \alpha_j)^{m_j} (e^{-it} - \overline{\alpha_j})^{m_j}}{\prod_{j=1}^q (e^{it} - \beta_j)^{d_j} (e^{-it} - \overline{\beta_j})^{d_j}}$$

with a constant  $c$ . Let  $m = m_1 + \dots + m_p$ ,  $d = d_1 + \dots + d_q$  and  $\varphi(z) = z^r \psi(z)$  where  $r = \max(0, m - d + 1)$ . Now,

$$\begin{aligned} \varphi(e^{it}) F(e^{it}) &= c e^{irt} \psi(e^{it}) \frac{\prod_{j=1}^p (e^{it} - \alpha_j)^{m_j} (e^{-it} - \overline{\alpha_j})^{m_j}}{\prod_{j=1}^q (e^{it} - \beta_j)^{d_j} (e^{-it} - \overline{\beta_j})^{d_j}} \\ &= c e^{irt} \prod_{j=1}^q \left( \frac{e^{it} - \beta_j}{1 - \overline{\beta_j} e^{it}} \right)^{d_j} \frac{\prod_{j=1}^p (e^{it} - \alpha_j)^{m_j} (1 - \overline{\alpha_j} e^{it})^{m_j} e^{-imt}}{\prod_{j=1}^q (e^{it} - \beta_j)^{d_j} (1 - \overline{\beta_j} e^{it})^{d_j} e^{-idt}} \\ &= c e^{i(d-m)t} e^{irt} \frac{\prod_{j=1}^p (e^{it} - \alpha_j)^{m_j} (1 - \overline{\alpha_j} e^{it})^{m_j}}{\prod_{j=1}^q (1 - \overline{\beta_j} e^{it})^{2d_j}} \\ &= c e^{it \max(d-m, 1)} \frac{\prod_{j=1}^p (e^{it} - \alpha_j)^{m_j} (1 - \overline{\alpha_j} e^{it})^{m_j}}{\prod_{j=1}^q (1 - \overline{\beta_j} e^{it})^{2d_j}} \end{aligned}$$

which implies that  $\varphi F \in \mathbb{H}_0^1$ . It follows from Lemma 2.2 that we have  $F = |f|^2$  with a suitable  $f \in H(\varphi)$ . Then for any integer  $k$ , we get

$$\begin{aligned} |c_k| &= | \langle F, e^{ikt} \rangle | &= | \langle f\overline{f}, e^{ikt} \rangle | \\ &= | \langle f e^{-ikt}, f \rangle | \\ &= | \langle (S^*(\varphi))^k f, f \rangle | \\ &= | \langle S^{*k}(\varphi) f, f \rangle |. \end{aligned}$$

Therefore

$$|c_k| \leq \|f\|_2^2 \omega_2(S^{*k}(\varphi)) = \|F\|_1 \omega_2(S^{*k}(\varphi)) = c_0 \omega_2(S^{*k}(\varphi)).$$

The proof is now complete.  $\square$

**Corollary 2.3** (Egerváry and Szász [4]). *Let  $F(e^{it}) = \sum_{j=-n+1}^{n-1} c_j e^{ijt}$  be a positive trigonometric polynomial ( $n \geq 2$ ). Then*

$$|c_k| \leq c_0 \cos \left( \frac{\pi}{\left[ \frac{n-1}{k} \right] + 2} \right) \text{ for } 1 \leq k \leq n-1.$$

*Proof.* We have :

$$\begin{aligned} F(e^{it}) &= c_{-n+1} e^{i(-n+1)t} + \dots + c_0 + \dots + c_{n-1} e^{i(n-1)t} \\ &= e^{(-n+1)it} \left( c_{-n+1} + \dots + c_0 e^{i(n-1)t} + \dots + c_{n-1} e^{2i(n-1)t} \right) \\ &= \frac{P(e^{it})}{Q(e^{it})} \end{aligned}$$

where  $P(e^{it}) = c_{-n+1} + \dots + c_0 e^{i(n-1)t} + \dots + c_{n-1} e^{2i(n-1)t}$  and  $Q(e^{it}) = e^{i(n-1)t}$ . In this case we have  $m = n-1$ ,  $d = 0$  and  $\varphi(z) = z^n$ . Then Theorem 2.1 implies that

$$|c_k| \leq c_0 \omega_2(S_n^{*k}).$$

But generally  $S_n^{*k}$  is unitarily equivalent to an orthogonal sum of shifts of smaller dimension, the largest dimension being  $s(k, n) + 1$  where  $s(k, n) = \left[ \frac{n-1}{k} \right]$ . Therefore  $\omega_2(S_n^{*k}) = \omega_2(S_{s(k, n)+1}^*) = \cos \frac{\pi}{s(k, n) + 2}$ . The same computation follows from [8]. Finally, this implies that

$$\begin{aligned} |c_k| &\leq c_0 \cos \frac{\pi}{s(k, n) + 2} \\ &= c_0 \cos \left( \frac{\pi}{\left[ \frac{n-1}{k} \right] + 2} \right). \end{aligned}$$

$\square$

**Remark 2.1.** The bound for  $c_1$  is due to Fejer (1915).

## 3. THE NUMERICAL RADIUS OF THE SHIFT COMPRESSION

**3.1. Preliminaries.** The spectral decomposition of the KMS matrix

$$K_n(\alpha) = \begin{pmatrix} 1 & \alpha & \cdots & \alpha^{n-1} \\ \alpha & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha \\ \alpha^{n-1} & \cdots & \alpha & 1 \end{pmatrix} = (\alpha^{|r-s|})_{r,s=1}^n ; 0 \leq \alpha < 1$$

is very well understood in the computational sense. For this reason, these matrices are often used as test matrices. It's shown in [7] page 69–72 that  $K_n(\alpha)$  is a Toeplitz matrix associated with the Poisson kernel  $P_\alpha(e^{it}) = \frac{1-\alpha^2}{|1-\alpha e^{it}|^2}$  and its eigenvalues are:

$$\lambda_k^{(n)} = P_\alpha(e^{it_k^{(n)}}), 1 \leq k \leq n$$

where  $t_k^{(n)}$  are the solutions of

$$(3.1) \quad p_n(\cos t) = \frac{\sin(n+1)t - 2\alpha \sin nt + \alpha^2 \sin(n-1)t}{\sin t} = 0.$$

The expression  $p_n(\cos t)$  is a polynomial of degree  $n$  in  $\cos t$  and it has  $n$  real distinct zeros  $\cos t_k^{(n)}$  for  $1 \leq k \leq n$  where :

$$0 < t_1^{(n)} < t_2^{(n)} < t_3^{(n)} < \cdots < t_n^{(n)} < \pi.$$

This implies that

$$\frac{1+\alpha}{1-\alpha} > \lambda_1^{(n)} > \lambda_2^{(n)} > \lambda_3^{(n)} > \cdots > \lambda_n^{(n)} > \frac{1-\alpha}{1+\alpha}.$$

The evaluation of the zeros  $t_k^{(n)}$  in explicit terms seems to be out of end. However, it is easy to show that they are separated by the quantities  $x_k = \frac{k\pi}{n+1}$ ,  $1 \leq k \leq n$ . Indeed, for  $1 \leq k \leq n$

$$p_n(\cos x_k) = (-1)^k 2\alpha(1 - \alpha \cos x_k)$$

and

$$\operatorname{sgn} p_n(\cos x_k) = (-1)^k.$$

Also we see by direct substitution that the latter equation holds for  $k=0$ , so that

$$0 < t_1^{(n)} \leq x_1 < t_2^{(n)} \leq x_2 < \cdots < t_n^{(n)} \leq x_n < \pi.$$

**Remark 3.1.** In the case where  $\alpha = 0$  in (3.1) we have  $t_k^{(n)} = x_k$ .

In the next proposition we give a better lower and upper bound for  $t_1^n$ .

**Proposition 3.1.** For each integer  $n \geq 2$ ;

$$(3.2) \quad \frac{2}{n+1} \arccos(\alpha) \leq t_1^{(n)} \leq \arccos(\alpha).$$

*Proof.* First, we note that

$$(3.3) \quad p_n(t) = \frac{2}{\sin t} \left( \sin \frac{(n+1)t}{2} - \alpha \sin \frac{(n-1)t}{2} \right) \left( \cos \frac{(n+1)t}{2} - \alpha \cos \frac{(n-1)t}{2} \right).$$

Since for all  $0 < t \leq \frac{\pi}{n+1}$ , we have

$$(3.4) \quad 0 < \frac{(n-1)t}{2} < \frac{(n+1)t}{2} \leq \frac{\pi}{2}$$

this implies that

$$\alpha \sin \frac{(n-1)t}{2} < \sin \frac{(n-1)t}{2} < \sin \frac{(n+1)t}{2}$$

then  $t_1^{(n)}$  is zero of

$$(3.5) \quad \cos \frac{(n+1)t}{2} = \alpha \cos \frac{(n-1)t}{2}.$$

Now if we suppose that  $t_1^{(n)} < \frac{2}{n+1} \arccos(\alpha)$  then  $\frac{(n+1)t_1^{(n)}}{2} < \arccos(\alpha)$  and  $\cos \frac{(n+1)t_1^{(n)}}{2} > \alpha \geq \alpha \cos \frac{(n-1)t_1^{(n)}}{2}$  which contradicts (3.5). Hence  $t_1^{(n)} \geq \frac{2}{n+1} \arccos(\alpha)$  holds.

From (3.5), we have

$$\cos t \cos \frac{(n-1)t}{2} - \sin t \sin \frac{(n-1)t}{2} = \alpha \cos \frac{(n-1)t}{2}$$

which implies that

$$(\cos t - \alpha) \cos \frac{(n-1)t}{2} = \sin t \sin \frac{(n-1)t}{2}$$

while from (3.4),  $\sin \frac{(n-1)t}{2}$  and  $\cos \frac{(n-1)t}{2}$  are both positive, therefore  $\cos t - \alpha$  is positive, which completes the proof.  $\square$

**Remark 3.2.** For a fixed  $0 \leq \alpha < 1$ , since  $P_\alpha(e^{it})$  is positive and decreasing on the interval  $[0, \pi]$ , then it is easy to obtain the sharp lower and upper bound of the largest eigenvalues of  $K_n(\alpha)$ :

$$1 \leq \lambda_1^{(n)} \leq \frac{1 - \alpha^2}{1 - 2\alpha \cos\left(\frac{2}{n+1} \arccos(\alpha)\right) + \alpha^2}.$$

Note that  $\lambda_1^{(n)}$  is also the numerical radius of  $K_n(\alpha)$ . This is due to the fact that the norm and numerical radius of a symmetric matrix coincides with its largest eigenvalue.

For  $0 \leq \alpha < 1$ , we denote by

$$s_n(\alpha) = \max \left\{ \frac{\alpha(\cos t_1^{(n)} - \alpha)t}{1 - 2\alpha \cos t_1^{(n)} + \alpha^2}; \frac{-\alpha(\cos t_n^{(n)} - \alpha)}{1 - 2\alpha \cos t_n^{(n)} + \alpha^2} \right\};$$

$$m_n(\alpha) = \max \left\{ \frac{|(1 + \alpha^2) \cos t_1^{(n)} - 2\alpha|}{1 - 2\alpha \cos t_1^{(n)} + \alpha^2}, \frac{-(1 + \alpha^2) \cos t_n^{(n)} + 2\alpha}{1 - 2\alpha \cos t_n^{(n)} + \alpha^2} \right\}$$

and

$$M_n(\alpha) = \max \left\{ \frac{(1 - 3\alpha^2) \cos t_1^{(n)} + 2\alpha^3}{1 - 2\alpha \cos t_1^{(n)} + \alpha^2}, \frac{-(1 + \alpha^2) \cos t_n^{(n)} + 2\alpha}{1 - 2\alpha \cos t_n^{(n)} + \alpha^2} \right\}.$$

**Proposition 3.2.** For each integer  $n \geq 2$  and  $0 \leq \alpha < 1$ , let

$$J_n(\alpha) = \begin{pmatrix} 0 & \alpha & \cdots & \alpha^{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \alpha \\ 0 & \cdots & \cdots & 0 \end{pmatrix};$$

then we have

$$\omega_2(J_n(\alpha)) = \omega_2(\mathcal{R}e(J_n(\alpha))) = s_n(\alpha).$$

*Proof.* First observe that

$$\begin{aligned}
\omega_2(\mathcal{Re}(J_n(\alpha))) &= \sup_{u=(u_0, \dots, u_{n-1}) \in \mathbb{C}^n, \|u\|=1} |\langle \mathcal{Re}(J_n(\alpha))u, u \rangle| \\
&= \frac{1}{2} \sup_{\sum_{l=0}^{n-1} |u_l|^2 = 1} \left| \sum_{1 \leq m \neq l \leq n-1} \alpha^{|l-m|} u_l \overline{u_m} \right| \\
&= \frac{1}{2} \sup_{\sum_{l=0}^{n-1} |u_l|^2 = 1} \sum_{1 \leq m \neq l \leq n-1} \alpha^{|l-m|} |u_l| |u_m| \\
&= \frac{1}{2} \sup_{\sum_{l=0}^{n-1} |u_l|^2 = 1} 2 \sum_{1 \leq m < l \leq n-1} \alpha^{|l-m|} |u_l| |u_m| \\
&= \sup_{\sum_{l=0}^{n-1} |u_l|^2 = 1} \left| \sum_{1 \leq m < l \leq n-1} \alpha^{l-m} u_l \overline{u_m} \right| = \omega_2(J_n(\alpha)).
\end{aligned}$$

We note that  $\mathcal{Re}(J_n(\alpha))$  is the Toeplitz matrix associated with the Toeplitz form

$$\frac{1}{2}(P_\alpha(e^{it}) - 1) = \frac{\alpha(\cos t - \alpha)}{1 - 2\alpha \cos t + \alpha^2} = g(t).$$

To complete the proof of the proposition, we can easily observe that if  $a$  and  $b$  are arbitrary real number and  $f(x)$  a Toeplitz form with  $\gamma_k^n$  as eigenvalues then the eigenvalues of  $a + bf(x)$  will be  $a + b\gamma_k^n$ . This shows that the eigenvalues of  $\mathcal{Re}(J_n(\alpha))$  are

$$\lambda'_k{}^{(n)} = \frac{1}{2}(\lambda_k^{(n)} - 1) = \frac{\alpha(\cos t_k^{(n)} - \alpha)}{1 - 2\alpha \cos t_1^{(n)} + \alpha^2}, \quad 1 \leq k \leq n.$$

Now, since  $g(t)$  is decreasing on the interval  $[0, \pi]$  and  $\mathcal{Re}(J_n(\alpha))$  is symmetric then

$$\begin{aligned}
\omega_2(J_n(\alpha)) &= \omega_2(\mathcal{Re}(J_n(\alpha))) \\
&= \max \left\{ |\lambda'_k{}^{(n)}|; 1 \leq k \leq n \right\} \\
&= \max \left\{ \lambda'_1{}^{(n)}; -\lambda'_n{}^{(n)} \right\} \\
(3.6) \quad &= \max \left\{ \frac{\alpha(\cos t_1^{(n)} - \alpha)}{1 - 2\alpha \cos t_1^{(n)} + \alpha^2}; \frac{-\alpha(\cos t_n^{(n)} - \alpha)}{1 - 2\alpha \cos t_n^{(n)} + \alpha^2} \right\} \\
&= s_n(\alpha)
\end{aligned}$$

where (3.6) is due to the fact that  $\cos t_n^{(n)}$  is nonpositive and by using Proposition 3.1.  $\square$

**Corollary 3.3.** For  $0 \leq \alpha < 1$ , we have

$$\omega_2(J_2(\alpha)) = \frac{\alpha}{2}.$$

*Proof.* This result is known, but it is interesting to notice that this result can also easily be obtained by using Proposition 3.2. Indeed

$$p_2(t) = \frac{\sin(3t) - 2\alpha \sin(2t) + \alpha^2 \sin t}{\sin t} = 4 \cos^2 t - 4\alpha \cos t + \alpha^2 - 1$$

Therefore, we obtain:  $\cos t_1^{(2)} = \frac{\alpha+1}{2}$  and  $\lambda'_1{}^{(2)} = \frac{\alpha}{2}$ . Now since  $Tr(J_2(\alpha)) = 0$  then we have  $\lambda'_2{}^{(2)} = -\frac{\alpha}{2}$ . This completes the proof.  $\square$



**Corollary 3.4.** For  $0 \leq \alpha < 1$ , we have

$$\omega_2(J_3(\alpha)) = \frac{\alpha(\sqrt{\alpha^2 + 8} - 3\alpha)}{4 + 2\alpha^2 - 2\alpha\sqrt{\alpha^2 + 8}}.$$

*Proof.* We have

$$p_3(t) = \frac{2}{\sin t} (\sin(2t) - \alpha \sin t) (\cos(2t) - \alpha \cos t)$$

this implies that  $\cos t_1^{(3)} = \frac{\alpha + \sqrt{\alpha^2 + 8}}{4}$ ,  $\cos t_2^{(3)} = \frac{\alpha}{2}$  and  $\cos t_3^{(3)} = \frac{\alpha - \sqrt{\alpha^2 + 8}}{4}$  then

$$\lambda_1'^{(3)} = \frac{\alpha(\cos t_1^{(3)} - \alpha)}{1 - 2\alpha \cos t_1^{(3)} + \alpha^2} = \frac{\alpha(\sqrt{\alpha^2 + 8} - 3\alpha)}{4 + 2\alpha^2 - 2\alpha\sqrt{\alpha^2 + 8}}$$

and

$$|\lambda_3'^{(3)}| = \frac{-\alpha(\cos t_3^{(3)} - \alpha)}{1 - 2\alpha \cos t_3^{(3)} + \alpha^2} = \frac{\alpha(\sqrt{\alpha^2 + 8} + 3\alpha)}{4 + 2\alpha^2 + 2\alpha\sqrt{\alpha^2 + 8}}.$$

The proof is complete.  $\square$

**Corollary 3.5.** For each integer  $n \geq 4$  and  $\alpha \leq \sqrt{\cos \frac{2\pi}{n+1}}$ , we have

$$\omega_2(J_n(\alpha)) = \frac{\alpha(\cos t_1^{(n)} - \alpha)}{1 - 2\alpha \cos t_1^{(n)} + \alpha^2}.$$

*Proof.* It follows from the hypothesis of Corollary 3.5 that

$$(3.7) \quad t_1^{(n)} \leq \frac{\pi}{n+1} \leq \arccos \sqrt{\frac{1+\alpha^2}{2}} \leq \arccos(\alpha) \leq t_n^{(n)} \leq \frac{n\pi}{n+1}$$

then

$$\lambda_1'^{(n)} = \frac{\alpha(\cos t_1^{(n)} - \alpha)}{1 - 2\alpha \cos t_1^{(n)} + \alpha^2}$$

and

$$|\lambda_n'^{(n)}| = -\lambda_n'^{(n)} = \frac{-\alpha(\cos t_n^{(n)} - \alpha)}{1 - 2\alpha \cos t_n^{(n)} + \alpha^2}$$

In view of the inequality (3.7),  $g(\arccos \alpha) = 0$  and the fact that  $g$  is decreasing in  $[0, \pi]$ , it suffices to prove that  $g(\frac{\pi}{n+1}) \geq |g(\frac{n\pi}{n+1})|$ . We have

$$\begin{aligned} g\left(\frac{\pi}{n+1}\right) - |g\left(\frac{n\pi}{n+1}\right)| &= \frac{\alpha(\cos \frac{\pi}{n+1} - \alpha)}{1 - 2\alpha \cos \frac{\pi}{n+1} + \alpha^2} - \frac{\alpha(\cos \frac{\pi}{n+1} + \alpha)}{1 + 2\alpha \cos \frac{\pi}{n+1} + \alpha^2} \\ &= \frac{2\alpha^2(2\cos^2 \frac{\pi}{n+1} - \alpha^2 - 1)}{(1 - 2\alpha \cos \frac{\pi}{n+1} + \alpha^2)(1 + 2\alpha \cos \frac{\pi}{n+1} + \alpha^2)} \\ &= \frac{2\alpha^2(\cos \frac{2\pi}{n+1} - \alpha^2)}{(1 - 2\alpha \cos \frac{\pi}{n+1} + \alpha^2)(1 + 2\alpha \cos \frac{\pi}{n+1} + \alpha^2)} \\ &\geq 0. \end{aligned}$$

The result follows.  $\square$

**3.2. The numerical radius of the compressed shift.** In this section, we will focus on the particular case where

$$\phi(z) = \left( \frac{z - \alpha}{1 - \bar{\alpha}z} \right)^n.$$

First, we notice some properties for the general case where  $\phi$  is a finite Blaschke product :  $\phi(z) = \prod_{j=1}^n \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}$ . For each  $\lambda$  in the unit disc, we define the evaluation functional  $k_\lambda \in \mathbb{H}^2$  by the requirement that  $f(\lambda) = \langle f, k_\lambda \rangle$ . Thus

$$k_\lambda(z) = \frac{1}{1 - \bar{\lambda}z}$$

and  $\{e_1, \dots, e_n\}$  the collection of functions of  $H(\phi)$  defined as follows :

$$e_1(z) = (1 - |\alpha_1|^2)^{\frac{1}{2}} \frac{1}{1 - \bar{\alpha}_1 z}$$

and

$$e_k(z) = (1 - |\alpha_k|^2)^{\frac{1}{2}} \frac{1}{1 - \bar{\alpha}_k z} \prod_{j=1}^{k-1} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}$$

for any  $k = 2, \dots, n$ .

It is known that  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $H(\phi)$  and with respect to this basis the matrix of  $S^*(\phi)$  is given by  $[a_{lk}]$ , where

$$a_{lk} = \begin{cases} \bar{\alpha}_l & \text{if } l = k \\ \sigma_l \sigma_{l+1} & \text{if } k = l + 1 \\ \sigma_l \sigma_k \prod_{j=l+1}^{k-1} (-\alpha_j) & \text{if } k > l + 1 \\ 0 & \text{unless} \end{cases}$$

and  $\sigma_k = (1 - |\alpha_k|^2)^{\frac{1}{2}}$ , for each  $1 \leq k \leq n$ .

Indeed, for  $k > l + 1$ , we have

$$\begin{aligned} \langle S^*(\phi)e_k, e_l \rangle &= \sigma_k \sigma_l \int_0^{2\pi} \frac{1}{1 - \bar{\alpha}_k e^{i\theta}} \frac{e^{-i\theta}}{1 - \alpha_l e^{-i\theta}} \prod_{j=l}^{k-1} \frac{e^{i\theta} - \alpha_j}{1 - \bar{\alpha}_j e^{i\theta}} \frac{d\theta}{2\pi} \\ &= \sigma_k \sigma_l \int_0^{2\pi} \frac{1}{1 - \bar{\alpha}_k e^{i\theta}} \frac{e^{-i\theta}}{1 - \alpha_l e^{-i\theta}} \frac{e^{i\theta} - \alpha_l}{1 - \bar{\alpha}_l e^{i\theta}} \prod_{j=l+1}^{k-1} \frac{e^{i\theta} - \alpha_j}{1 - \bar{\alpha}_j e^{i\theta}} \frac{d\theta}{2\pi} \\ &= \sigma_k \sigma_l \int_0^{2\pi} \frac{1}{1 - \bar{\alpha}_k e^{i\theta}} \frac{1}{1 - \bar{\alpha}_l e^{i\theta}} \prod_{j=l+1}^{k-1} \frac{e^{i\theta} - \alpha_j}{1 - \bar{\alpha}_j e^{i\theta}} \frac{d\theta}{2\pi} \\ &= \sigma_k \sigma_l \prod_{j=l+1}^{k-1} (-\alpha_j). \end{aligned}$$

Using the same scheme as before, we prove easily that  $\langle S^*(\phi)e_{k+1}, e_k \rangle = \sigma_k \sigma_{k+1}$  and  $\langle S^*(\phi)e_k, e_l \rangle = 0$  if  $k < l$ .

Finally

$$\begin{aligned} \langle S^*(\phi)e_k, e_k \rangle &= \sigma_k^2 \int_0^{2\pi} \frac{1}{1 - \bar{\alpha}_k e^{i\theta}} \frac{e^{-i\theta}}{1 - \alpha_k e^{-i\theta}} \frac{d\theta}{2\pi} \\ &= \sigma_k^2 \langle K_{\alpha_k}, \frac{z}{1 - \bar{\alpha}_k z} \rangle \\ &= \bar{\alpha}_k \end{aligned}$$

In the sequel of the paper,  $\phi$  denotes the finite Blaschke product with unique zero  $\alpha$ :

$$\phi(z) = \left( \frac{z - \alpha}{1 - \bar{\alpha}z} \right)^n.$$

$S^*(\phi)$  gets the following matricial representation:

$$\begin{pmatrix} \bar{\alpha} & \sigma & -\sigma\alpha & \cdots & \cdots & \sigma(-\alpha)^{n-2} \\ 0 & \bar{\alpha} & \sigma & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -\sigma\alpha \\ \vdots & & & \ddots & \bar{\alpha} & \sigma \\ 0 & \dots & \dots & \dots & 0 & \bar{\alpha} \end{pmatrix}$$

where  $\sigma = 1 - |\alpha|^2$ .

**Proposition 3.6.** For  $0 \leq \alpha < 1$ , we have

$$\omega_2(\mathcal{Re}(S^*(\phi))) = m_n(\alpha).$$

*Proof.* First, notice that where  $\alpha = 0$ , then

$$\mathcal{Re}(S^*(\phi)) = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

In this case the eigenvalues are  $\cos \frac{k\pi}{n+1}$ , for  $k = 1, \dots, n$ . For the proof there are many references, we refer the reader for example to [7] page 67 or [2] page 35, therefore  $\omega_2(\mathcal{Re}(S^*(\phi))) = \cos \frac{\pi}{n+1} = m_n(0)$ . Then we can limit our study to the case  $\alpha \neq 0$ . Now, notice that

$$\mathcal{Re}(S^*(\phi)) = \frac{1 - \alpha^2}{2\alpha} \begin{pmatrix} -\frac{2\alpha^2}{1 - \alpha^2} & \alpha & \dots & \alpha^{n-1} \\ \alpha & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha \\ \alpha^{n-1} & \dots & \alpha & -\frac{2\alpha^2}{1 - \alpha^2} \end{pmatrix}.$$

Here  $\mathcal{Re}(S^*(\phi))$  is the Toeplitz matrix associated with the Toeplitz form:

$$\frac{1 - \alpha^2}{2\alpha} (P_\alpha(e^{it}) - 1 - \frac{2\alpha^2}{1 - \alpha^2}) = \frac{(1 + \alpha^2) \cos t - 2\alpha}{1 - 2\alpha \cos t + \alpha^2} = h(t).$$

Since  $h(t)$  is monotonic on  $[0, \pi]$ , thus with the same argument that in the proof of Proposition 3.2, we may assume that:

$$\begin{aligned} \omega_2(\mathcal{Re}(S^*(\phi))) &= \max \left\{ \frac{|(1 + \alpha^2) \cos t_1^n - 2\alpha|}{1 - 2\alpha \cos t_1^n + \alpha^2}, \frac{-(1 + \alpha^2) \cos t_n^n + 2\alpha}{1 - 2\alpha \cos t_n^n + \alpha^2} \right\} \\ &= m_n(\alpha). \end{aligned}$$

This completes the proof.  $\square$

**Theorem 3.7.** Let  $\phi(z) = \left(\frac{z - \alpha}{1 - \bar{\alpha}z}\right)^n$  with  $\alpha \in \mathbb{C}$  and  $|\alpha| < 1$ .

- (1) The numerical radius of  $S^*(\phi)$  is independent from the argument of  $\alpha$  and for  $0 \leq \alpha < 1$  the numerical range of  $S^*(\phi)$  is symmetric with respect to the real axis.
- (2) For  $n \geq 2$ , we have

$$m_n(|\alpha|) \leq \omega_2(S^*(\phi)) \leq M_n(|\alpha|).$$

*Proof.* For  $\alpha \neq 0$ , and for  $t = \arg(\alpha)$ , we have:

$$\begin{aligned}
 \omega_2(S^*(\phi)) &= \sup_{\|u\|_2=1} |\langle S^*(\phi)u, u \rangle| \\
 &= \sup_{\sum_{l=0}^{n-1} |u_l|^2=1} \left| \bar{\alpha} - \frac{1 - |\alpha|^2}{\alpha} \sum_{1 \leq m < l \leq n-1} (-\alpha)^{l-m} u_l \bar{u}_m \right| \\
 &= \sup_{\sum_{l=0}^{n-1} |u_l|^2=1} \left| |\alpha| - \frac{1 - |\alpha|^2}{|\alpha|} \sum_{1 \leq m < l \leq n-1} (-\alpha)^{l-m} u_l \bar{u}_m \right| \\
 &= \sup_{\sum_{l=0}^{n-1} |u_l|^2=1} \left| |\alpha| - \frac{1 - |\alpha|^2}{|\alpha|} \sum_{1 \leq m < l \leq n-1} (-|\alpha|)^{l-m} e^{ilt} u_l e^{imt} \bar{u}_m \right| \\
 (3.8) \quad &= \sup_{\sum_{l=0}^{n-1} |v_l|^2=1} \left| |\alpha| - \frac{1 - |\alpha|^2}{|\alpha|} \sum_{1 \leq m < l \leq n-1} (-|\alpha|)^{l-m} v_l \bar{v}_m \right|.
 \end{aligned}$$

The last equality implies that the numerical radius of  $S^*(\phi)$  is independent from the argument of  $\alpha$ . Hence we can suppose that  $0 < \alpha < 1$ . Now assume that  $z$  is in  $W(S^*(\phi))$ , then there is  $u = (u_0, \dots, u_{n-1})$  a unit vector in  $\mathbb{C}^n$  such that

$$\begin{aligned}
 z &= \langle S^*(\phi)u, u \rangle \\
 &= \alpha - \frac{1 - \alpha^2}{\alpha} \sum_{1 \leq m < l \leq n-1} (-\alpha)^{l-m} u_l \bar{u}_m
 \end{aligned}$$

and

$$\begin{aligned}
 \bar{z} &= \alpha - \frac{1 - \alpha^2}{\alpha} \sum_{1 \leq m < l \leq n-1} (-\alpha)^{l-m} u_m \bar{u}_l \\
 &= \langle S^*(\phi)\bar{u}, \bar{u} \rangle.
 \end{aligned}$$

This implies that  $\bar{z}$  is in  $W(S^*(\phi))$  and (1) holds.

As remarked before, we can restrict our study to the case where  $0 < \alpha < 1$  and from (3.8) we have:

$$\begin{aligned}
 \omega_2(S^*(\phi)) &\leq \alpha + \frac{1 - \alpha^2}{\alpha} \sup_{\sum_{l=0}^{n-1} |u_l|^2=1} \left| \sum_{1 \leq m < l \leq n-1} \alpha^{l-m} u_l \bar{u}_m \right| \\
 &= \alpha + \frac{1 - \alpha^2}{\alpha} \omega_2(J_n(\alpha)) \\
 &= M_n(\alpha).
 \end{aligned}$$

On the other hand, it is obvious to note that  $\omega_2(\mathcal{R}eT) \leq \omega_2(T)$  for each bounded operator  $T$ , an application of the Proposition 3.6 completes the proof of (2).  $\square$

**Corollary 3.8.** *For  $n = 2$  and  $0 \leq \alpha < 1$ ; we have:*

$$\omega_2(S^*(\phi)) = \frac{1 + 2\alpha - \alpha^2}{2}.$$

**Corollary 3.9.** *For each integer  $n \geq 2$ ;*

$$\omega_2(S_n^*) = \cos \frac{\pi}{n+1}.$$

In the last two corollaries, the results are due to the fact that both quantities  $m_n$  and  $M_n$  coincide.

Acknowledgements: The author would like to express his gratitude to Gilles Cassier for his help and his good advices and Alfonso Montes Rodriguez as well for advices in the translation of this work..

## REFERENCES

- [1] C. Badea and G. Cassier, Constrained von Neumann inequalities, Adv. Math. 166 (2002), no. 2, 260–297.
- [2] A. Böttcher and S. Grudsky, Spectral properties of banded Toeplitz matrices. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005.
- [3] G. Cassier, I. Chalendar, and B. Chevreau, New examples of contractions illustrating membership and non membership in The classe  $A_{n,m}$ , Acta Sci. Math (Szeged) 64 (1998), 701–731.
- [4] E.V. Egev  r  y and O. Sz  asz, Einige Extremalprobleme in Bereiche der trigonometrischen Polynomen, Math. Z. 27 (1928), 641–652.
- [5] H. L. Gau and P. Y. Wu, Numerical range and Poncelet property, Taiwanese J. Math. 7 (2003), no. 2, 173–193.
- [6] H. L. Gau and P. Y. Wu, Numerical range of  $S(\phi)$ . Linear and Multilinear Algebra 45 (1998), no. 1, 49–73.
- [7] U. Grenander and G. Szeg  , Toeplitz forms and their applications. California Monographs in Mathematical Sciences University of California Press, Berkeley-Los Angeles 1958.
- [8] K.E. Gustafson, D.K.M. Rao: Numerical Range, Springer, New York, 1997.
- [9] U. Haagerup and P. de la Harpe, The numerical radius of a nilpotent operator on a Hilbert space, Proc. Amer. Math. Soc. 115(1992), 371–379.
- [10] P. R. Halmos, A Hilbert space problem book. Second edition. Graduate Texts in Mathematics, 19. Encyclopedia of Mathematics and its Applications, 17. Springer-Verlag, New York-Berlin, 1982.
- [11] R. A. Horn and C. R. Johnson, Topics in matrix analysis. Cambridge University Press, Cambridge, 1991.
- [12] M. Kac, W. L. Murdock and G. Szeg  , On the eigenvalues of certain Hermitian forms. J. Rational Mech. Anal. 2, (1953). 767–800.
- [13] B. Sz.-Nagy and C. Foias, Harmonic analysis of operator on Hilbert space, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akad  miai Kiado, Budapest, 1970.
- [14] W. F. Trench, Interlacement of the even and odd spectra of real symmetric Toeplitz matrices. Linear Algebra Appl. 195 (1993), 59–68.

‡.INSTITUTE CAMILLE JORDAN, OFFICE 107 UNIVERSITY OF LYON1, 43 BD NOVEMBER 11, 1918, 69622-VILLEURBANNE, FRANCE.

E-mail address: ‡gaaya@math.univ-lyon1.fr